Long Contest Editorial

November 12, 2015

Moscow International Workshop ACM ICPC, MIPT, 2015

A. Too Rich

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Observation

Denote T total amount of dollars we have. Obtaining P dollars using the most number of coins is the same as taking $T - P$ using the least number of coins and leaving them out. In the following we discuss the problem of representing S using minimal amount of coins.

A. Too Rich

Example (A simpler case)

Consider a set of denominations $d_1 < \ldots < d_k$ such that every denomination divides the previous one: $d_{i+1} \vdots d_i$ for all $i \in [1; k-1]$. Can we come up with an easy solution for the same problem?

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Greedy algorithm for the simpler case

In this case a greedy algorithm works: take maximal amount of d_k-dollar coins such that the sum does not exceed S, then take maximal amount of d_{k-1}-dollar coins, and so on. If the total amount of money taken this way is S, then the representation is minimal, otherwise no representation is possible.

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Proof for the greedy algorithm

Suppose that $c_1 d_1 + \ldots + c_j d_j \geqslant d_{j+1}$ for some integer non-negative c_j. Then we can choose integer non-negative c'_j such that $c'_j \leqslant c_j$ and $c'_1 d_1 + \ldots + c'_j d_j = d_{j+1}$. This can be done by induction: take maximal possible amount of d_j-dollar coins, and represent the rest using first $j - 1$ denominations (the rest amount is divisible by d_j).
Now, consider any representation of $P = c_1 d_1 + \ldots + c_k d_k$. If c_k is not maximal possible, choose a subset of smaller coins with sum d_k and replace them with a single coin; repeat until the sum of smaller coins becomes less than d_k. So on for smaller coins.

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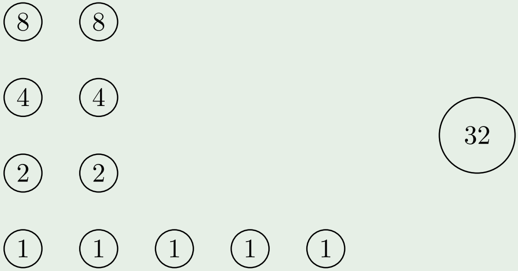
Example (Choose a subset with sum of exactly 32)



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2 2

1 1 1 1 1

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However, the greedy approach can be slightly modified to work here. Suppose that on some step of the greedy algorithm the maximal number of d_j-dollar coins that we can take is X. Then, there is a minimal representation such that the number X' of d_j-dollar coins we take is at least $X - 1$, because if X' is at most $X - 2$, we can always replace a subset of smaller coins with a total amount of $2d_j$ with two d_j-dollar coins; the existence of this subset if proved similarily to greedy solution analysis.

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Thus, the algorithm that recursively tries X and $X - 1$ for the number of largest coins will always give an optimal answer. Without any optimizations this performs $\sim 2^9$ operations per test, which works fast enough.

B. Count $a \times b$

Let $f(n)$ be the number of pairs $0 \leqslant a, b < n$ such ab is not divisible by n, and $g(n) = \sum_{d|n} f(d)$. Find $g(n)$.

B. Count $a \times b$

Let's start with $f(n)$. $f(n) = n^2 - h(n)$, where $h(n)$ is the number of pairs $0 \leqslant a, b < n$ such that $ab \vdots n$.

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Factorize n: $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Chinese remainder theorem implies that $h(n)$ is multiplicative: $h(n) = h(p_1^{\alpha_1}) \dots h(p_k^{\alpha_k})$.

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Find $h(p^\alpha)$. For $0 \leqslant a < p^\alpha$ let $d(a)$ be the maximal power of p dividing a (set $d(0) = \alpha$ by definition). For any given a and b:

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$$ab \vdots p^\alpha \iff d(a) + d(b) \geqslant \alpha.$$

For any $0 \leqslant k \leqslant \alpha$, the number of a's such that $d(a) \geqslant k$ is exactly $p^{\alpha-k}$. Thus, we obtain the formula:

$$h(p^\alpha) = \sum_{k=0}^{\alpha-1}((p^{\alpha-k} - p^{\alpha-k-1})p^k) + p^\alpha = \alpha p^\alpha - (\alpha - 1)p^{\alpha-1}.$$

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Both $s_2(n)$ and $H(n)$ can be computed easily given factorization of n. It can be found straghtforwardly in $O(\sqrt{n})$, with posslble speed-up to $O(\sqrt{n}/\log n)$ using precomputed prime tables up to $\sqrt{n}$.

C. Play a game

We are given a string s and a set of forbidden strings A. Two players play a game: if at the beginning of one's turn the current string is empty or belongs to A, the player loses immediately, otherwise, he can erase a symbol either from the beginning of from the end of the string. Find winning player for several substrings of s as starting strings.

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- otherwise, $w_{l,r} = W$ if one of $w_{l+1,r}$ or $w_{l,r-1}$ is L, otherwise, $w_{l,r} = L$.

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Example

Let $s = abacaba$, $A = \{b, bac, cab\}$

The table of $w_{l,r}$ looks as follows:

$l \backslash r$	0	1	2	3	4	5	6	7
0	L							
1		L						
2			L					
3				L				
4					L			
5						L		
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3. Lose because $w_{l+2,r}$ and $w_{l,r-2}$ are loses (e.g. $[1;6)$)

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It can easily be shown that in all other cases $w_{l,r}$ is indeed equal to $w_{l+1,r-1}$.

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Knowing that, we will do the following: store two adjacent diagonals of the table, and gradually move them to the up and to the right while processing all the cases the elements change and answering queries off-line. We assume that we know all occurences of elements of A as substrings of s.

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For each query and each occurence store the index of diagonal it is concerned.

- When answering a query, simply access the diagonal's element (we assume that is has been maintained correctly)
- When processing an occurence, change the element of the diagonal to L, and the elements immediately to the up and to the right to W

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Let T be the total number of occurences of elements of A as substrings of s. It can be shown that the number of times situation 3 arises is $O(T)$. Therefore, the total number of events occuring during the "sweep-line diagonal" process is $O(T)$, and its time complexity is $O(T)$.

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We use Aho-Corasick to find all occurences in time $O(n + L + T)$.

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The number of different lengths is maximal if lengths are 1, 2, ..., and is $O(\sqrt{L})$, which implies the statement.

This concludes the analysis of the problem. The resulting solution has complexity $O(L + n\sqrt{L})$.

D. Pipes selection

We are given an array of non-negative integers with sum s. Let k_x be total number of segments with sum x. For every x from 1 to s find $\lfloor \frac{k_x+1}{2} \rfloor$-th lexicographically smallest segment with sum x (segments are ordered by left end, then by right end).

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Let p_n be the sum of first n elements, and let q_x be the number of such n that $p_n = x$. Construct polynomials $A(x) = \sum_{i=0}^{s} q_i x^i$ and $B(x) = \sum_{i=0}^{s} q_{s-i} x^i$. Define $C(x) = A(x)B(x) = \sum_{i=0}^{2s} c_i x^i$.

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Sqrt-decomposition helps. Let's divide the array into t blocks of equal size.

For j-th block with beginning l_j and end r_j, construct polynomial $B_j(x) = \sum_{i=l_j}^{r_j} q_{s-i}x^i$, and define $C_j(x) = A(x)B_j(x) = \sum_{i=0}^{2s} c_{j_i}x^i$.

The number of segments with sum x which left end lies in segment $[l_j; r_j]$ is exactly $c_{j_{s+x}}$.

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- First, find the block which contains the beginning of the sought segment by simply iterating the blocks from left to right (this requires comparisons of lex number with $c_{j_{s+x}}$.
- Then, iterate over elements inside the block to find the actual segment (this doesn't require any knowledge about $C_j(x)$)

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To achieve optimum choose $t \sim \sqrt{n/\log s}$, for a total complexity of $O(s\sqrt{n \log s})$.
In practice, FFT has significantly higher intrinsic constant factor, which means that in order to balance it out, t should be slightly lower than the theoretic optimum.

E. Rebuild

Given a sequence of points in the plane, build circles centered at each point such that circles which are centered at consecutive points are tangent, and also the first and the last circles are tangent. Minimize total area of circles.

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Let d_i be the distance between i-th point and $(i+1)$-th point, and d_n be the distance between the first and the last point. Restate the problem: we have to choose radii x_i such that $x_1 + x_2 = d_1, \ldots,$ $x_n + x_1 = d_1$, while minimizing $\sum x_i^2$.

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  Finally, substituting expressions for x_i, obtain $\sum x_i^2 = ax_1^2 + bx_1 + c$ for some real a, b, c.
  Minimizing a quadratic function on a segment is trivial: if global minimum $x_0 = -\frac{b}{2a}$ belongs to $[L; R]$, then x_0 is the answer, otherwise one of the segment ends L, R is the answer.

F. Almost sorted array

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Let us try to make array non-decreasing, and then repeat the procedure for the reversed array.
Suppose that we have erased a_i. The resulting array is non-decreasing if first $i - 1$ elements are sorted, last $n - i$ elements are sorted, and $a_{i-1} \leqslant a_{i+1}$ (if $i = 1$ or $i = n$, this condition is redundant). Find the longest sorted prefix and suffix, then try to erase each element. This makes for a simple $O(n)$ solution.

G. Dancing Stars on Me

Given a set of points with integer coordinates, determine if it coincides with the set of vertices of a regular polygon.

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To show this, consider three consecutive vertices A, B, C of the regular n-gon. Observe that vector $\overline{BC}$ is the vector $\overline{AB}$ rotated by $2\pi/n$. Since both vectors have integer coordinates, we conclude that $\cos(2\pi/n)$ and $\sin(2\pi/n)$ are both rational. The only $n \geqslant 3$ satisfying this is $n = 4$.

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Checking that four given points are at vertices of a square is trivial.

H. Partial Tree

We want to build a tree on n vertices. For a vertex of degree i we get score d_i. Maximize total score (over all vertices).

H. Partial Tree

Which degree sequences $d_1, \ldots, d_n$ correspond to trees on n vertices? Trivial necessary coniditions are $d_i \geqslant 1$ and $\sum d_i = 2(n-1)$.

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Thus we have to solve a variety of the backpack problem: given cost for an item of every weight from 1 to $n-1$, choose n items with total weight of $2(n-1)$ and maximal possible cost. For convenience, we substract 1 from all weights, so the total weight becomes $n-2$.

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Start from the set of n items of weight 0, then replace them with heavier items one by one. Denote dp_w the maximal cost of a set with total weight w obtained this way. By definition, $dp_0 = nf(0)$, $dp_w = \max_{k=1}^{w} dp_{w-k} + f(k) - f(0)$. The answer is dp_{n-2}. This yields an $O(n^2)$ solution.

I. Chess puzzle

We are given a rectangular board, where some cells have fixed colors (black or white), while some haven't. We have to color all non-colored cells. We get 1 point for every pair of cells (x_1, y_1) and (x_2, y_2) if:

- $|x_1 - x_2| = a$, $|y_1 - y_2| = b$ $(a, b > 0)$
- cells (x_1, y_1) and (x_2, y_2) are of different colors

Find a coloring that maximizes total score, if there are several colorings, choose lexicographically minimal.

I. Chess puzzle

Let's ignore lex-min requirement for now. How to build any optimal coloring?

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If we add edges between pairs of cells with $|x_1 - x_2| = a$, $|y_1 - y_2| = b$, we obtain a bipartite graph. We can construct a convenient partition: first a rows are in the first part, next a rows are in the second part and so on.

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If $n = m = 5$, $a = 2$, the partition looks as follows:

1	1	1	1	1
1	1	1	1	1
2	2	2	2	2
2	2	2	2	2
1	1	1	1	1

Flip the colors of all cells in the second part. Now we have to maximize number of adjacent pairs with the *same* color.

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Any $S - T$ cut corresponds to coloring (S's part — black, T's part — white), and its capacity is exactly the number of points of different color. Minimal $S - T$ cut corresponds to coloring with maximal number of same-colored adjacent pairs.

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Any sufficiently fast algorithm for max-flow (e.g. Dinic) will allow us to build some minimal cut.

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If color of the current cell is `W`, then we should try to change it to B. Assume that colors of all earlier cells are fixed by adding edges from source/to sink.

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Suppose that changing is possible. Add edge between the cell and source/sink (depending on whether the cell's color was flipped or not). The value of min-cut should not increase; equivalently, it should be impossible to push one unit of flow after adding the edge. It means that the cell and sink/source should not be connected in the residual network.

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To sum up, we have to add edges to the network and for each vertex remember whether it is reachable from the source, and whether sink is reachable from it using edges of residual network.

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To sum up, we have to add edges to the network and for each vertex remember whether it is reachable from the source, and whether sink is reachable from it using edges of residual network. To do this, after adding every edge run DFS from the cell adjacent to it; visibility markings should not be cleared between runs. This process takes $O(n^2)$ total time, which means complexity depends entirely on the max-flow algorithm used.

J. Chip Factory

Given an array s_i, find

$$\max_{i,j,k \text{ — distinct indices}} (s_i + s_j) \oplus s_k$$

.

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Once we have fixed the l-th bit, we choose $(l-1)$-th bit according to the same reasoning, and so on.

At every given moment, several greatest bits of s_k are fixed. Next bit of s_k depends on whether we can choose k (different from i and j) so that a prefix of s_k matches our preference.

J. Chip Factory

s: $i = 1, j = 3$

00100_2

10110_2

00101_2

00111_2

01011_2

$s_i + s_j = 11101_2$

$s_k = ?????$

J. Chip Factory

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3 possible s_k match prefix.

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2 possible s_k match prefix.

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00111_2

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$s_k = 000??_2$

No possible s_k match prefix, have to choose 2-nd bit otherwise.

J. Chip Factory

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No possible s_k match prefix (note that s_3 matches but k has to be different from i and j).

J. Chip Factory

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$$s_i + s_j = 11101_2$$

$$s_k = 00100_2$$

Maximal possible $(s_i + s_j) \oplus s_k$ is $11101_2 \oplus 00100_2 = 11001_2 = 25$.

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Total working time of this solution is $O(n^2 l)$.

K. Maximum spanning forest

Consider a region of rectangular grid. We have to answer n queries "add edges with weight c between all pairs of adjacent points inside a rectangle", and find the weight of maximal spanning forest built on the points inside a region.

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Compress the coordinates: let x_i be the sorted sequence of different x's appearing as a border coordinate of a query rectangle; similarly, construct y_i.

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Compress the coordinates: let x_i be the sorted sequence of different x's appearing as a border coordinate of a query rectangle; similarly, construct y_i.

Call a rectangle *elementary* if its x-borders are adjacent elements of x_i, and same for y-borders.

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Compress the coordinates: let x_i be the sorted sequence of different x's appearing as a border coordinate of a query rectangle; similarily, construct y_i.

Call a rectangle *elementary* if its x-borders are adjacent elements of x_i, and same for y-borders. Denote $R_{i,j}$ $[x_i; x_{i+1}) \times [y_j; y_{j+1})$.

Clearly, there are $O(n^2)$ elementary rectangles.

K. Maximum spanning forest

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All edges of the grid lie either inside of an elementary rectangle, or connect two points of adjacent elementary rectangles.

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At any moment, all edges inside an elementary rectangle have the same weight (if we consider only the heaviest of multiple edges), and all edges between points of two adjacent rectangles have the same weight.

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Introduce the following arrays:

- $w_{i,j}$ — the weight of edges inside the elementary rectangle $R_{i,j}$
- $h_{i,j}$ — the weight of edges between $R_{i,j}$ and $R_{i+1,j}$
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For each query update entries of arrays which correspond to edges lying inside the query rectangle. Each update takes $O(n^2)$ time.

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In every elementary rectangle all points become merged into a single component, for a total weight of (number of points $-$ 1) $\cdot\ w_{i,j}$.

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In every elementary rectangle all points become merged into a single component, for a total weight of (number of points $- 1) \cdot w_{i,j}$.
After that, we can consider each elementary rectangle a single vertex. Thus, adding in-between edges is reduced to building MST of a simple graph with $O(n^2)$ vertices and edges. That is, every query can be answered in $O(n^2 \log n)$ time, for a total $O(n^3 \log n)$ complexity solution.

L. House Building

Given a set of $1 \times 1 \times 1$ cubes on rectangular grid lying on the ground in several towers, determine the outer area of the construction.

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Thus, just iterate over all adjacent pairs of towers, there is only linear ($O(nm)$) number of them. This solution is $O(nm)$.

M. Security Corporations

We are given a set of lines in the plane, no three of them share a point. Choose minimal number c and assign an index from $[1; c]$ to every intersection of two lines in such a way that every neighbouring intersections on the same line have different indices.

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If there are three lines forming a triangle, then it's easy to show that some intersections form a triangle as well, so $c \geqslant 3$.

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It's easy to see that the algorithm constructs a correct 3-coloring. Moreover, it uses minimal number of colors in cases when $c < 3$. Its complexity is $O(n^2 \log n)$, the hardest part being sorting of points.